REVIEW OF BASIC ANALYSIS ON THE REAL LINE

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The main reference for these notes is Rudin's Principles [R1976].

1. Sequences and series

Let $\mathbb{R}$ denote the real numbers, $\{a_n\}_{n=0}^{\infty}$ is the typical sequence.

1. a_n converges to a (write $a_n \longrightarrow a$) as $n \to \infty$,
iff $|a_n - a| \longrightarrow 0$, as $n \to \infty$;
iff $\forall \epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$ s.t.

$$n > N \Longrightarrow |a_n - a| \leq \epsilon.$$

Exercise 1.1. Show that if a_n converges to a, then $\{a_n\}$ is bounded. And conversely, if $\{a_n\}$ is bounded (above) and $a_n \leq a_{n+1}$ (monotone increasing), then $\{a_n\}$ converges (Hint: take the supremum of $\{a_n\}$).

2. $\{a_n\}_{n=0}^{\infty}$ is a Cauchy sequence
iff $\forall \epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$ s.t.

$$m > n > N \Longrightarrow |a_m - a_n| \leq \epsilon.$$

Exercise 1.2. Show that, in $\mathbb{R}$, a sequence converges iff it is Cauchy. (This is known as the *Cauchy criterion*).

3. $\{a_{n_k}\}_{k=0}^{\infty}$ with $n_k \in \mathbb{N}$ and $n_k < n_{k+1}$ is a subsequence of $\{a_n\}_{n=0}^{\infty}$. A sublimit of $\{a_n\}$ is the limit of a convergent subsequence.

Exercise 1.3. A sequence converges iff every subsequence converges to the same sublimit.

4. The series $\sum_{n=0}^{\infty} a_n$ converges iff the sequence of partial sums $\{\sum_{n=0}^{k} a_n\}_{k=0}^{\infty}$ converges. The series $\sum_{n=0}^{\infty} a_n$ converges absolutely iff $\sum_{n=0}^{\infty} |a_n|$ converges. We write $\sum_{n=0}^{\infty} |a_n| < \infty$.

Exercise 1.4. Show that if $\sum a_n$ converges, then $a_n \longrightarrow 0$. Hint: use the Cauchy criterion.

Exercise 1.5. Show that $\sum_{n=1}^{\infty} 1/n$ diverges even though $1/n \to 0$ as $n \to \infty$. Hint: $1/3 > 1/2$ and $1/5, 1/6, 1/7 > 1/8$ etc...

Exercise 1.6. Show that if $\sum a_n$ converges absolutely, then it converges. Hint: use the Cauchy criterion.

5. The limit inferior $L := \liminf_{n\to\infty} a_n$ always exists and can be defined in many different ways.

Exercise 1.7. Show that the following definitions of L are equivalent:

(1) $L = \sup\{h : \#\{n : a_n < h\} < \infty\}$.
(2) Let E be the set of all possible sublimits of $\{a_n\}$. Then $L = \min E$.
(3) $L = \sup_k \inf_{n\geq k} a_n$.
(4) L is the unique number such that
  (a) $(\forall\epsilon > 0)(\exists N = N(\epsilon) \in \mathbb{N})$ s.t. $n > N \Longrightarrow a_n \geq L - \epsilon$.
  (b) $(\forall\epsilon > 0)(\forall n \in \mathbb{N})(\exists k > n)$ s.t. $a_n < L + \epsilon$.

Hint: Label L in (1) as L_1, etc...then show that $L_1 \leq L_2$ and $L_2 \leq L_1$ etc...

Exercise 1.8. State a similar exercise for the limit superior.

6. The Geometric Series.

Exercise 1.9. Show that for $x \in \mathbb{R} \setminus \{1\}$ and $N = 0, 1, 2, \dots,$

$$1 + x + x^2 + x^3 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

Hint: Multiply both sides by $(1 - x)$ and use "telescoping".

Deduce that $\sum_{n=0}^{\infty} x^n$ converges iff $|x| < 1$, and determine the sum in that case.

7. The Comparison Test: If for some $N_0 \in \mathbb{N}$, $|a_n| \leq c_n$ for $n \geq N_0$, and $\sum c_n$ converges, then $\sum a_n$ converges as well.

Exercise 1.10. Prove the Comparison Test using the Cauchy criterion and the Triangle Inequality.

8. The Root Test: Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

- $\alpha < 1 \implies \sum a_n$ converges.
- $\alpha > 1 \implies \sum a_n$ diverges.
- $\alpha = 1 \implies$ inconclusif.

Exercise 1.11. Prove the Root Test by picking β with $\alpha < \beta < 1$ and comparing to a geometric series.

Exercise 1.12. Show that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Deduce a "Ratio Test" from this.

9. Power Series:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$\{a_n\}_{n=0}^{\infty}$ is the sequence of coefficients. Form $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ as above. Then $R = 1/\alpha$ is the Radius of Convergence of the power series.

Exercise 1.13. Use the Root Test to show that $\sum a_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$.

10. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=0} a_{\sigma(n)}$ is a rearrangement of $\sum_{n=0}^{\infty} a_n$.

Exercise 1.14. Show that if $\sum a_n$ converges absolutely, then for every rearrangement σ we have $\sum a_n = \sum a_{\sigma(n)}$. Hint: define $\phi : \mathbb{N} \to \mathbb{N}$ as

$$\phi(N) = \inf \{n : \{1, \ldots, N\} \subset \{\sigma(1), \ldots, \sigma(n)\}\}.$$

Then $\phi(N) \geq N$ and for $m > \phi(N)$ the partial sums satisfy

$$\left| \sum_{n=1}^{m} a_n - \sum_{n=1}^{m} a_{\sigma(n)} \right| \leq \sum_{n>N+1} |a_n|.$$

11. Addition and scalar multiplication of convergent series results in convergent series and the new series can be obtain doing the operations term-by-term. Multiplication is not so simple. One way is the so-called Cauchy product: given series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ consider $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Exercise 1.15. If $\sum a_n$ converges absolutely and $\sum a_n$, $\sum b_n$ converge respectively to A and B, then the Cauchy product $\sum c_n$ as above converges to AB. See [R1976] Theorem 3.50.

Cauchy Products arise naturally in the context of power series. Consider the following manipulation which is entirely *formal*:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n,k} a_n b_k x^{n+k} = \sum_{j=0}^{\infty}\left(\sum_{n=0}^{j} a_n b_{j-n}\right) x^j = \sum_{j=0}^{\infty} c_j x^j$$

where we changed variables from (n, k) to (n, j) so that $j = n + k$.

12. A (formal) double sum $\sum_{n,m}^{\infty} c_{n,m}$ converges absolutely iff

$$S := \sup_{N,M} \sum_{\substack{n=0,\ldots,N \\ m=0,\ldots,M}} |c_{n,m}| < \infty.$$

Exercise 1.16. If a double sum $\sum_{n,m}^{\infty} c_{n,m}$ converges absolutely, then

$$\sum_{n=0}^{\infty}\left(\sum_{m=0} |c_{n,m}|\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0} |c_{n,m}|\right)$$

and both sides equal S.

Deduce from this that for each n, $\sum_{m=0}^{\infty} c_{n,m}$ converges to say s_n; and for each m, $\sum_{n=0}^{\infty} c_{n,m}$ converges to say t_m; moreover $\sum_{n=0}^{\infty} s_n$ and $\sum_{m=0}^{\infty} t_m$ also converge, and in fact converge to the same limit, which we are then justified in denoting as $\sum_{n,m=0}^{\infty} c_{n,m}$.

Hint: Split the quadrant $\mathbb{N}\times\mathbb{N}$ into a rectangle, two half-strips and one rectangular sector using the lines $\{(n,m) : n = N\}$ and $\{(n,m) : m = M\}$. Then estimate the partial sums

$$\left|\sum_{n=0}^{N} S_n - \sum_{m=0}^{M} T_m\right|$$

where $S_n = \sum_{m=0}^{\infty} |c_{n,m}|$ and $T_m = \sum_{n=0}^{\infty} |c_{n,m}|$.

2. Continuity and uniform convergence

13. Let X, Y be metric spaces. A function $f : X \to Y$ is uniformly continuous iff $\exists\ \eta : [0,\infty) \to [0,\infty)$ homeomorphism s.t.

$$x_1, x_2 \in X \Longrightarrow d_Y(f(x_1), f(x_2)) \le \eta(d_X(x_1, x_2)).$$

Exercise 2.1. Show that if f is uniformly continuous then $\forall\epsilon > 0\ \exists\delta = \delta(\epsilon) > 0$ s.t.

$$d_X(x_1, x_2) \le \delta \Longrightarrow d_Y(f(x_1), f(x_2)) \le \epsilon.$$

Show that the converse holds if $X = \mathbb{R}$ but not in general, e.g. consider $f(n) = 2^{2^n}$ on $\mathbb{Z}$.

Remark 2.2. f isLipschitz continuous if it is uniformly continuous with $\eta(t) = Ct$ for some $C > 0$; f is α-Hölder continuous if it is uniformly continuous with $\eta(t) = O(t^\alpha)$ as $t \to 0$, for some $0 < \alpha \le 1$; and f is Dini-continuous if it is uniformly continuous with $\int_0 \frac{\eta(t)}{t} dt < \infty$.

Facts (don't prove, but reread):

• Let X, Y be metric spaces . X compact $\Longrightarrow$ every continous $f : X \to Y$ is uniformly continuous.

• The uniform limit of a sequence of continuous functions is continuous.

• If $f_n \to f$ uniformly on a bounded interval I, then

$$\lim_{n\to\infty} \int_I f_n(x)dx = \int_I \lim_{n\to} f_n(x)dx = \int_I f(x)dx$$

and if the partial sums of $\sum f_n$ converge uniformly on I, then

$$\sum_{n=0}^{\infty} \int_I f_n(x)dx = \int_I \sum_{n=0}^{\infty} f_n(x)dx.$$

References

[R1976] Rudin, W. *Principles of Mathematical Analysis*. McGraw-Hill, Third Edition, 1976.

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